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Resonant acoustic oscillations with damping: small rate theory

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A gas in a tube is excited by a reciprocating piston operating at or near a resonant frequency. Damping is introduced into the system by two means: radiation of energy from one end of the tube and rate dependence of the gas. These define a lumped damping coefficient. It is shown that in the small rate limit the signal in the periodic state suffers negligible distortion in one travel time, and hence its propagation according to acoustic theory is valid. The shape of the signal is determined by a nonlinear ordinary differential equation. The small rate condition provides a test of the applicability of the theory to given experimental conditions. When there is no damping, shocks are a feature of the flow for frequencies in the resonant band. For a given amount of damping an upper bound on the piston acceleration which ensures shockless motion is given. The resonant band is analysed for both damped and undamped cases. The predictions of the theory are compared with experiment.

1. Introduction

In this paper we discuss the periodic vibrations which result when a column of gas in a Kundt tube is driven by a piston oscillating at or near a resonant frequency. The basic experimental observations are well documented (Saenger & Hudson 1960). When the piston frequency is in a band about a resonant frequency, the amplitude of the response is markedly higher than the piston amplitude and shock waves appear in the flow. These phenomena have been extensively investigated in the recent literature (Betchov 1958; Saenger & Hudson 1960; Chu & Ying 1963; Chester 1964; Mortell 1971 *a*, *b*; Collins 1971). Nevertheless, there are associated phenomena which need further investigation, and some aspects of the various analyses which need clarification. Two questions which are resolved concern the range of validity of the usual modifications of acoustic theory used previously, and the existence of a critical amount of damping which ensures a shockless motion.

The basis for the analysis given here is the fact that the motion of the gas can be represented, to first order in the amplitudes, as the superposition of two non-interacting simple waves travelling in opposite directions (see Mortell & Varley

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1970). This implies that the travel time of a component wave in the tube is determined by its own amplitude, and then any distortion of a signal is self-induced. The essence of this approximation, which is the first term in a regular perturbation expansion, is that while the amplitudes remain small, $|u| \ll a_0$, the acceleration $|\partial u/\partial t|$ in the gas is not restricted. (Here, u is the particle velocity and a_0 is the equilibrium sound speed.) This yields a linear representation for amplitudes, but a nonlinear representation for characteristics. A nonlinear theory such as this has been used for many years in gasdynamics for disturbances generated by the passage of a single progressing wave (see Whitham 1952).

We consider the motion of a gas in a tube which is driven by a piston oscillating at one end. The other end is 'partially open' in the sense that it is allowed to radiate energy into the adjacent medium. We seek the time-periodic response of the gas to these boundary conditions. Using the simple-wave representation the problem of calculating the shape of the signal at a boundary is reduced to finding solutions of a nonlinear functional difference equation. The signal may be distorted as it travels, its shape at any particle in the body of the gas being calculated from the simple-wave representation. In the 'small rate' or 'small acceleration' limit $|\partial u/\partial t| \ll a_0^2/L$ (where L is a typical length of the medium), the functional *difference* equation can be reduced to a nonlinear ordinary *differential* equation describing the shape of the signal in the *periodic* state. Further, in this limit, the distortion of the time-periodic signal in one travel time is negligible and propagation according to linear acoustic theory is valid. All previous investigations of this problem have implicitly used the small rate approximation where the signal shape is determined by an ordinary differential equation. However, it is evident that, for a given maximum piston displacement ϵ, the small rate limit restricts the allowed values of the applied frequency and hence the allowed resonant mode (given by $n = 1, 2, 3, \ldots$). For an applied amplitude $\epsilon = 0{\cdot}0147$ (Sturtevant 1972, private communication) the restriction is that $n \ll 2$, while for $\epsilon = 0{\cdot}0018$ (Saenger & Hudson 1960) it requires that $n \ll 6$. Thus the restriction implied by the small rate limit is surprisingly strong. For larger amplitudes or higher resonant modes the functional difference equation must therefore be analysed without any approximations. This is done in a forthcoming paper.

There are two basic phenomena in the model used here: shocks due to nonlinearity, and damping which can prevent shocks. When there is no damping present, linear theory predicts an unbounded amplitude in the periodic state for certain discrete (resonant) frequencies. On the other hand, nonlinear theory predicts a bounded signal, which contains shocks, in a band about the resonant frequencies. Since shocks act as a dissipative mechanism they allow a balance of energy. This role of nonlinearity seems to be well-understood. To date, the most comprehensive investigation of the effect of damping on resonant motions is due to Chester (1964). He investigated the effects of compressive viscosity and boundary-layer friction on the motion.

In the theoretical analysis given here, damping is introduced into the system by allowing energy to radiate from the end of the tube, and through the rate dependence of the gas. We then show that damping introduced in this manner can prevent the occurrence of shocks in the flow; i.e. for a given piston motion there is a critical level of damping above which the gas motion is continuous. (This does not answer the question raised by Chester (1964) specifically concerning the existence of a critical amount of boundary-layer damping.) There is experimental evidence to support the prediction. B. Sturtevant, at the California Institute of Technology, has carried out experiments in which a hole is made in the closed end of the tube. He found that for the particular conditions of an experiment there is a critical ratio of the area of the hole to the area of the tube end at which shocks disappear.

In § 3 there is a complete analysis of the resonant band for both the damped and undamped cases for a quite general periodic forcing function h. Previous investigations have considered only the case when h was a pure harmonic. The results are obtained by an examination of the integral curves of the governing differential equation, using a condition on the mean flow to fix the shock position. For the undamped case explicit analytical results are given for the shape of the resulting signal, the edge of the resonant band, the position of a shock and the shock strength. For the damped motion, qualitative results are found analytically, while quantitative results are found numerically. In § 5 there is a comparison between theory and experiment. The introduction of damping improves the agreement and provides an adequate description of such gross features of the flow as the maximum or minimum pressure, or the shock strength.

2. Formulation

A column of gas, of length L in some reference (equilibrium) state, is contained in a pipe. One end of the pipe is closed while at the other end there is an oscillating piston. If pressure and density are measured from their values in the reference state (p_0, ρ_0) with the associated sound speed a_0, then in terms of the non-dimensional variables $a_0 u$, $\rho_0 a_0 p$, and $\rho_0 \rho$ and Lx and $La_0^{-1} t$ the governing equations in Lagrangian form are

$$[(1+e)^{-1}]_t - u_x = 0 \tag{2.1}$$

and

$$u_t + p_x = 0, \tag{2.2}$$

where e ($= \rho - 1$) is the condensation, γp the excess pressure ratio and u the non-dimensional particle velocity. The equation of state of a polytropic gas in these variables is

$$\gamma p = (1+e)^\gamma - 1. \tag{2.3}$$

The end $x = 0$ is considered to be 'near rigid' in the sense that we allow for the possibility of radiation of energy through this end of the tube, but do not consider the case when it is open. A boundary condition of this nature has been discussed by Mortell & Varley (1970). Across the boundary at $x = 0$ both pressure and velocity are continuous, and so the disturbance must be compatible with the homogeneous boundary condition

$$u(0, t) = -ip(0, t), \tag{2.4}$$

where i/γ ($\geqslant 0$) is the impedance of the interface. Here the essential assumption is that the disturbance outside the tube is generated by the passage of a simple wave. Note that $i = 0$ corresponds to a rigid end and $i = \infty$ to an open end. We examine the small amplitude, time-periodic response of the gas, governed by equations (2.1)–(2.3), to the boundary condition (2.4) at $x = 0$ and a periodic piston displacement at $x = 1$ of the form $\epsilon h(\omega t)$. The amplitude of the displacement is ϵ ($\ll 1$), and the period of h is normalized so that $h(y+1) = h(y)$. Then the piston velocity at $x = 1$ is

$$u(1,t) = \epsilon\omega h'(\omega t) = H(\omega t). \tag{2.5}$$

Since h is periodic, integration of (2.5) yields

$$\int_0^1 H(s)\,ds = 0. \tag{2.6}$$

Equations (2.1)–(2.3) are nonlinear and admit discontinuous solutions. However, it has been shown by Mortell & Seymour (1972*b*) that for time-periodic motions, be they continuous or discontinuous, the mean pressure and velocity do not vary from particle to particle. By choosing as the reference pressure p_0 the constant mean of the periodic state, conditions (2.4) and (2.6) imply that the means of u and p are zero. The actual value of p_0 can be determined only from an initial-value problem.

2.1. *Equation for the periodic motion*

A representation derived by Mortell & Varley (1970) is used to reduce the nonlinear boundary-value problem defined by (2.1)–(2.5) to a nonlinear difference, or functional, equation. For a more restricted class of problems, this functional equation may be further reduced to a nonlinear ordinary differential equation which determines the shape of the periodic signal on one boundary.

It is convenient to reformulate equations (2.1)–(2.3) in terms of the Riemann invariants and characteristic curves of the system. Upon defining

$$c(e) = \int_0^e a(s)\,(1+s)^{-1}\,ds = e[1+\tfrac{1}{2}(M-1)\,e+O(e^2)],$$

where
$$a^2(e) = (1+e)^2\,dp/de = 1+2Me+O(e)^2$$

and $M = \frac{1}{2}(\gamma+1)$, equations (2.1)–(2.3) define the Riemann invariants

$$2f(\beta) = u-c = u-p+O(e^2) \tag{2.7}$$

and
$$-2g(\alpha) = u+c = u+p+O(e^2). \tag{2.8}$$

The associated characteristics are given by

$$\left.\frac{dx}{dt}\right|_\alpha = a(e), \quad \left.\frac{dx}{dt}\right|_\beta = -a(e). \tag{2.9}$$

When only one component of the motion is excited, (2.7)–(2.9) admit two exact solutions, simple waves, which correspond to $f \equiv$ constant and $g \equiv$ constant. When both components of the motion are excited there is in general an interaction between α waves, moving to the right, and β waves, moving to the left. However, it has been shown by Mortell & Varley (1970) that to first order, in the limit of *small amplitudes*, the waves do not interact as they pass through each other in the body of the gas. By this is meant that to first order the trajectory of an α wave is determined only by the signal it carries and is not influenced by the β waves through which it passes. Thus the motion of the gas may be represented as the superposition of two non-interacting simple waves. Then (2.7) and (2.8) imply that to first order

$$e = p = -f(\beta)-g(\alpha), \quad u = f(\beta)-g(\alpha), \tag{2.10}$$

while (2.9) integrate to give

$$\alpha/\omega = t-x-Mxg(\alpha), \quad \beta/\omega = t+x-1+M(x-1)f(\beta), \tag{2.11}$$

where we have parametrized α and β by $\alpha = \omega t$ on $x = 0$ and $\beta = \omega t$ on $x = 1$. The representations (2.11) imply that different amplitudes have different speeds and hence different travel times in the tube. Thus an immediate consequence of the nonlinear theory is the existence of resonant frequency bands rather than discrete resonant frequencies.

Upon using the boundary conditions (2.4) and (2.5), g is eliminated from (2.10) and (2.11) to yield the nonlinear functional difference equation to determine the signal f on the boundary $x = 1$:

$$f(\eta)-kf(s) = H(\eta), \tag{2.12}$$

where

$$\eta = s+2\omega+\omega M(1+k)f(s). \tag{2.13}$$

In (2.12) and (2.13), $k = (1-i)/(1+i)$ is the *reflexion coefficient* at $x = 0$, where g is related to f by

$$g(\phi+\omega+\omega Mf(\phi)) = kf(\phi). \tag{2.14}$$

The governing differential equations (2.1)–(2.3) and the boundary conditions (2.4) and (2.5) have been reduced in the small amplitude limit to the functional difference equation (2.12) and (2.13). We now seek solutions to (2.12) and (2.13) which, like the piston motion, have unit period. Further, as a consequence of the representations (2.10) and (2.11), the boundary conditions (2.4) and (2.5) and the fact that u and p have zero mean over any period, f and g must satisfy

$$\int_0^1 f(s)\,ds = \int_0^1 g(s)\,ds = 0. \tag{2.15}$$

Since M is the ratio of second-order to first-order elastic constants, linear theory is recovered from (2.12) and (2.13) by formally setting $M = 0$ to yield

$$f(\eta)-kf(\eta-2\omega) = H(\eta). \tag{2.16}$$

When $k = 1$ there are no solutions of (2.16) with unit period; that is, when

$$\omega = \omega_n = \tfrac{1}{2}n \quad (n = 1, 2, 3, \ldots). \tag{2.17}$$

These are the linear resonant frequencies. Ultimately we shall consider the time-periodic response of the system to frequencies near to those defined by (2.17) and consequently define

$$\omega = \omega_n(1+\delta) \quad (|\delta| < 1/n). \tag{2.18}$$

Then in terms of

$$F(y) = f(y)+\delta/b, \tag{2.19}$$

where
$$b = \tfrac{1}{2}M(1+k)(1+\delta) \quad (= O(1)), \tag{2.20}$$
(2.12) and (2.13) become
$$F(\eta) - kF(s) = G(\eta) \tag{2.21}$$
and
$$\eta = s + n + nbF(s), \tag{2.22}$$
where
$$G(\eta) = \mu\delta/b + H(\eta), \quad \mu = 1-k. \tag{2.23}$$
Definition (2.19) now implies that the zero-mean condition (2.15) on f is replaced by
$$\int_0^1 F(s)\,ds = \delta/b. \tag{2.24}$$

The approximations used to derive (2.21) and (2.22) are the small amplitude assumption $|f|, |g| \ll 1$ and the fact that the impedance of the interface at $x = 0$ is near zero, so that $0 \leqslant k \leqslant 1$. This latter assumption is required since whenever $1+k = O(e)$ the next correction to the characteristics (at order e^2) is no longer negligible. For example, if the end of the tube is *open* ($k = -1$) the *nonlinear* approximation (2.11) to the characteristics leads to a *linear* difference equation which has no bounded periodic solution at a resonant frequency. Consequently, for a problem involving an open end (or 'nearly' open) the approximations (2.11) for α and β must be improved. In fact, the approximations for α and β must contain terms in f^2 and g^2, so that now the motion in the tube is determined by the cubic term in the equation of state, with a resulting amplitude of $O(\epsilon^{\frac{1}{2}})$ (see Collins 1971; Seymour & Mortell 1972). Note further that the difference equation (2.21) and (2.22) together with (2.19) determine the shape of the signal function f only on $x = 1$. The velocity u and pressure p are subsequently calculated at any particle x in the tube from the representations (2.10) and (2.11). This is particularly important when there is significant distortion of a wave in one travel time.

If we now make the additional *small rate assumption*
$$|nF'| \ll 1,$$
(2.22) implies that
$$F(s) = F(\eta - n) - nbF(\eta - n)\,F'(\eta - n)\,[1 + O(nF')]. \tag{2.25}$$
Upon using (2.25), and since we seek solutions with unit period, the difference equation (2.21) and (2.22) can then be approximated by the nonlinear ordinary differential equation
$$\nu F(\eta)\,F'(\eta) + \mu F(\eta) = G(\eta) \quad (0 \leqslant \eta \leqslant 1), \tag{2.26}$$
together with
$$F(\eta+1) = F(\eta), \tag{2.27}$$
where $\nu = nbk$. The small rate condition and the definition of F give the further restriction $|\delta| \ll 1/n$. Thus (2.26) is valid in the small rate limit only for periodic motions at frequencies in the neighbourhood of linear resonant frequencies. In contrast to this, the difference equation (2.21), and (2.22), was derived with no restriction on the applied rates; it is valid for non-periodic phenomena (see Mortell & Varley 1970; Mortell & Seymour 1972*a, b*), and the applied frequency is not restricted to lie near a linear resonant frequency.

Since the small amplitude restriction requires that $|f| \ll 1$, the small rate condition implies that the differential equation (2.26) is a good approximation to the difference equation only when n is at most of $O(1)$ as $|f| \to 0$. In addition, the representations (2.11) for the characteristics imply that in the small rate limit there will be no appreciable distortion of the wave form in one period, since

$$\left|\frac{f(\beta)-f(\omega(t+x-1))}{f(\beta)}\right| \leqslant |\omega M f'(\theta)|,$$

where θ lies between β and $\omega(t+x-1)$, and $0 \leqslant x \leqslant 1$. (Of course there will be a cumulative distortion of the signal until the periodic motion has been set up. This is not described by (2.26).) Hence, *in the small rate limit*, (2.10) and (2.11) can be replaced by the linear acoustic representation

$$-p = f(\omega[t+x-1])+g(\omega[t-x]), \quad u = f(\omega[t+x-1])-g(\omega[t-x]), \quad (2.28)$$

so that, when the periodic motion has evolved, nonlinearity is of primary importance in determining the shape of the signal functions, but it is of secondary importance in determining how the signals propagate.

We point out that the small rate condition $|nF'| \ll 1$, which is necessary for the validity of both the differential equation (2.26) and the representation (2.28), is quite restrictive for usual experimental values of the parameters. The results of § 3 show that for $\epsilon = 0{\cdot}0147$, used by Sturtevant (private communication), this restriction implies $n \ll 2$; for $\epsilon = 0{\cdot}0018$, used by Saenger & Hudson (1960), the restriction yields $n \ll 6$. In a sequel to this paper we shall examine periodic motions with no restrictions on the rates when some of the ideas introduced here are used to analyse the functional difference equation (2.12) and (2.13) directly.

It may be of interest to note that equation (2.26) arises in other physical situations. It is a generalization of the equation which describes the motion of a viscously damped pendulum under a constant external moment, and also occurs in the study of the pull-out torque of a synchronous motor, see Stoker (1950) or Minorsky (1962). If one were to write $i = \epsilon^{\frac{1}{2}}i_1+\epsilon i_2+\dots$ or $k = 1+\epsilon^{\frac{1}{2}}k_1+\dots$, then (2.26) could be derived using the regular perturbation scheme given in Mortell (1971*b*). This latter derivation, unlike that given here, has an explicit assumption on the amount of damping present. The analysis of (2.26), under the restriction (2.24), and the physical interpretation of the results constitute the remainder of this paper.

3. Determination of the periodic signal

Here we analyse the integral curves of the differential equation (2.26) for various ranges of the parameters μ and δ and then use these to construct the signal F of the periodic motion. When we have constructed F over one period, possibly by a composition of integral curves, it is continued periodically by (2.27). The signal function F may then have *discontinuities* representing a time-periodic motion in the pipe containing shocks. The discontinuities in F arise in satisfying the mean condition (2.24). Acoustic theory allows discontinuities of either compression or rarefaction with no restriction on their strengths. However, to be physically acceptable a jump in a gas must be compressive. Here we consider only piston velocities which have three zeros over one period, and then discontinuous solutions of (2.26), which satisfy the mean condition (2.24), additionally satisfy the appropriate weak-shock relations. This is not strictly necessary within the acoustic approximation.

If $S(x)$ is the arrival time at x of a weak shock travelling in the negative x direction and $\beta^{\pm}(x)$ are the wavelets immediately ahead of and behind the shock, then the weak-shock relations imply that

$$dS/dx = -1 - \tfrac{1}{2}M[f(\beta^+) + f(\beta^-)]. \quad (3.1)$$

A similar relation gives the speed of shocks moving to the right. However, since in the periodic state there is negligible distortion, $\beta^{\pm}$ are independent of x and (3.1) can be integrated to give the travel time of a shock from $x = 1$ to $x = 0$. The boundary condition (2.14) then implies that the total travel time for the shock to return to $x = 1$ is

$$T = 2 + \tfrac{1}{2}M(1+k)[f(\beta^+) + f(\beta^-)]. \quad (3.2)$$

Since the periodicity requirement is that $T = n/\omega$, equation (3.2) implies that

$$f(\beta^+) + f(\beta^-) = -2\delta/b. \quad (3.3)$$

On using (2.19), the definition of F, equation (3.3) becomes

$$F(\beta^+) + F(\beta^-) = 0. \quad (3.4)$$

The condition that only compressive shocks are allowed then requires that $F(\beta^+) > 0$.

On the other hand, integration of (2.26) over one period, assuming a discontinuity at $\eta = \beta$, yields

$$\tfrac{1}{2}\nu[F^2(\beta^+) - F^2(\beta^-)] + \mu \int_0^1 F(s)\,ds = \int_0^1 G(s)\,ds. \quad (3.5)$$

Conditions (2.6), (2.23) and (2.24) then imply that (3.4) must hold at the discontinuity. Thus a solution of (2.26), containing a discontinuity, which satisfies the mean condition (2.24) and the restriction $F(\beta^+) > 0$ will necessarily satisfy the weak-shock relations. Thus a shock is fitted into the solution by satisfying the mean condition. This analysis is for one shock per period of the piston, which gives n shocks in the tube at any time.

3.1. *Special case of no damping* $\mu = 0$

When the boundary at $x = 0$ is rigid and there is no radiation of energy through it, $i = \mu = 0$ and equation (2.26) is greatly simplified. It can then be integrated completely and the signal function, the width of the resonant band and the shock strength for these frequencies can be determined analytically in terms of the parameters of the problem. Further, the transition from a discontinuous motion inside the band to a continuous one outside is exhibited explicitly.

We wish to distinguish between the integral curves of equation (2.26) and the signal F which must additionally satisfy the mean condition (2.24). An integral curve is denoted by $Z(\eta)$. Notice that while F is defined only for $0 \leqslant \eta \leqslant 1$ and is then continued periodically, if F is continuous it must coincide with an integral curve Z which is both continuous and periodic for $-\infty < \eta < \infty$. Conversely, a continuous periodic integral curve Z with unit period which satisfies the mean condition (2.24) is the required signal function F. When such an integral curve exists it is unique. When, for a particular frequency, no such curve exists, F is discontinuous and is composed of distinct integral curves.

For the case $\mu = 0$, the appropriate differential equation is

$$nbZ(\eta)\,Z'(\eta) = H(\eta). \tag{3.6}$$

There is no loss of generality in choosing the origin so that

$$H(0) = H(\eta_1) = H(1) = 0,$$

where $0 < \eta_1 < 1$, with $H'(0) = H'(1) > 0$ and $H'(\eta_1) < 0$. Then in the η, Z plane the points $A_0 = (0,0)$, $A_2 = (1,0)$ and $B_1 = (\eta_1, 0)$ are isolated singular points. A_0 and A_2 are saddle points, while B_1 is a centre. Using (2.5), the integral curves which are defined for all η are given by

$$Z^{\pm}(\eta, \delta) = \pm\,[(\epsilon/M)\{h(\eta)-h(0)\}+C(\delta)]^{\frac{1}{2}}, \tag{3.7}$$

where $C(\delta) = Z^2(0,\delta)$. These solutions are periodic, with unit period for all η. The two separatices through A_0, $Z_0^{\pm}(\eta)$, correspond to $C(\delta) = 0$ and these, by the periodicity of h, connect the saddle points A_0 and A_2.

Since $Z_0^{+}(\eta) > 0$ for $0 < \eta < 1$, any solution $Z(\eta)$ with $Z(0) > 0$ is periodic in η with $Z(\eta) > Z_0^{+}(\eta)$ and therefore Z satisfies

$$\int_0^1 Z(s)\,ds > \int_0^1 Z_0^{+}(s)\,ds. \tag{3.8}$$

Consequently the mean condition (2.24) implies that for an applied frequency $\omega = \frac{1}{2}n(1+\delta)$ such that

$$\frac{\delta}{b} > \int_0^1 Z_0^{+}(s)\,ds \tag{3.9}$$

there exists a unique, *continuous*, periodic solution $Z_\delta(\eta) = Z^{+}(\eta,\delta)$. The positive constant $C(\delta)$ is chosen so that $Z_\delta(\eta)$ satisfies the mean condition (2.24). A similar analysis shows there are also continuous periodic solutions $Z_\delta(\eta) = Z^{-}(\eta,\delta)$ for

$$\frac{\delta}{b} < \int_0^1 Z_0^{-}(s)\,ds < 0.$$

For frequencies such that

$$\int_0^1 Z_0^{-}(s)\,ds < \frac{\delta}{b} < \int_0^1 Z_0^{+}(s)\,ds \tag{3.10}$$

no single integral curve will satisfy the mean condition (2.24) and the signal function will necessarily be *discontinuous*. The shock condition together with the fact that only compressive shocks are allowed then implies that the signal function F can only be constructed from the separatices $Z_0^{\pm}(\eta)$ with just one shock per period. The position of the shock at $\eta = \eta_s$ is chosen to satisfy the mean condition. The signal function F is then given by

$$F(\eta) = \begin{Bmatrix} Z_0^{+}(\eta) & 0 \leqslant \eta < \eta_s, \\ Z_0^{-}(\eta) & \eta_s < \eta \leqslant 1. \end{Bmatrix} \tag{3.11}$$

The range of frequencies, defined by (3.10), for which the signal is *discontinuous*, is called the *resonant band*. If (3.10) is solved for δ the resonant band is given explicitly by $\delta^- < \delta < \delta^+$, where

$$\delta^{\pm} = \frac{\pm(\epsilon M)^{\frac{1}{2}}\bar{h}(1)}{1 \mp (\epsilon M)^{\frac{1}{2}}\bar{h}(1)} \tag{3.12}$$

and
$$\bar{h}(\eta) = \int_0^{\eta} [h(s)-h(0)]^{\frac{1}{2}}\, ds.$$

Notice that the amplitude of the response of the gas to an applied signal of $O(\epsilon)$ is of $O(\epsilon^{\frac{1}{2}})$ and that the width of the resonant band $2(\epsilon M)^{\frac{1}{2}}\bar{h}(1)\,[1-\epsilon M\bar{h}^2(1)]^{-1}$ is also of $O(\epsilon^{\frac{1}{2}})$. The position of the shock, $\eta = \eta_s$, is given implicitly by

$$\bar{h}(\eta_s) = \tfrac{1}{2}\bar{h}(1) + \delta/2(\epsilon M)^{\frac{1}{2}}(1+\delta), \tag{3.13}$$

where the shock strength is

$$Z_0^+(\eta_s) - Z_0^-(\eta_s) = 2(\epsilon/M)^{\frac{1}{2}}\{h(\eta_s)-h(0)\}^{\frac{1}{2}}. \tag{3.14}$$

On using (3.12) and (3.13), as $\delta \to \delta^+$, $\eta_s \to 1$, while as $\delta \to \delta^-$, $\eta_s \to 0$. Thus by (3.14), as $\delta \to \delta^{\pm}$, the shock strength tends to zero. The limiting solutions, when $\delta = \delta^{\pm}$, are given by $F(\eta) = Z_0^{\pm}(\eta)$. Then the signal F is continuous but has a *discontinuous slope* at $\eta = 0, 1$. The resonant band is not symmetrically situated about the linear resonant frequencies $\omega = \omega_n$, since $|\delta^+| > |\delta^-|$, by (3.12).

The above results are particularly simple for the important special forcing function $h(\eta) = -\cos 2\pi\eta$. The edges of the resonant band are

$$\delta^{\pm} = \pm 2(2\epsilon M)^{\frac{1}{2}}/[\pi \mp 2(2\epsilon M)^{\frac{1}{2}}].$$

The shock is located at η_s, where

$$\cos(\pi\eta_s) = -\pi\delta/2(1+\delta)(2\epsilon M)^{\frac{1}{2}},$$

and the shock strength is $2(2\epsilon/M)^{\frac{1}{2}}\sin \pi\eta_s$. Note that $F'(\eta_1) = 0$ and thus when $\omega = \omega_n$, $\eta_s = \eta_1 = \frac{1}{2}$, i.e. the maximum pressure equals the pressure immediately ahead of the shock, and the pressure immediately behind the shock is the minimum pressure.

3.2. $\mu_c < \mu < 1$

When μ is non-zero the positions of the singular points in the η, Z plane now depend on both μ and δ, and B_1 is no longer a centre. We consider the variations in (μ, δ) in two parts. Here we show that for a given forcing function $H(\eta)$ there is a critical amount of damping $\mu = \mu_c$ such that for $\mu > \mu_c$ the signal function is continuous for all frequencies. In § 3.3 we fix $\mu < \mu_c$ and consider variations in δ which will define the resonant band.

If we assume that the zeros of $H(\eta)$ satisfy the conditions described in § 3.1, the singular points of the equation

$$\nu Z(\eta)\, Z'(\eta) + \mu Z(\eta) = G(\eta), \tag{3.15}$$

where $G(\eta)$ is defined by (2.23), are the points $(\theta_i, 0)$, $i = 0, 1, 2$, such that $G(\theta_i) = 0$. (The periodicity of H ensures that $\theta_2 = \theta_0 + 1$.) We label them $A_0(\mu, \delta)$, $A_2(\mu, \delta)$ and $B_1(\mu, \delta)$, where $A_0(\mu, 0) = (0, 0)$, $A_2(\mu, 0) = (1, 0)$ and $B_1(\mu, 0) = (\eta_1, 0)$.

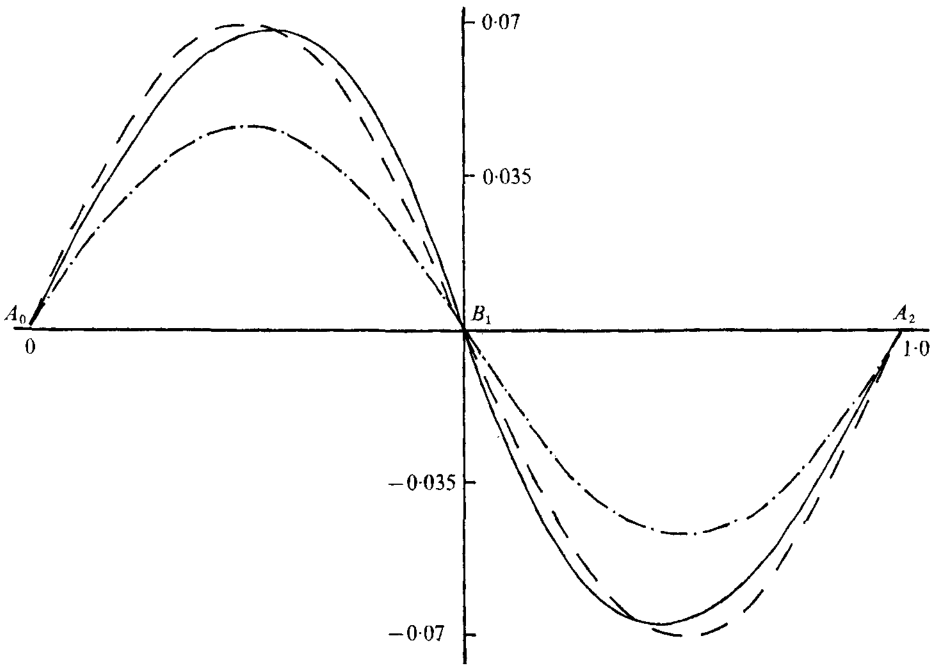

FIGURE 1. Highly damped case; linear theory a good approximation.
$\epsilon = 0{\cdot}0147$, $\delta = 0$, $i = 0{\cdot}50$. ——, $F(\eta)$; ---, $F_L(\eta) = G(\eta)/\mu$; —·—, $G(\eta)$.

Labelling them in this way is consistent with the notation of the previous section and ensures that A_0 and A_2 are again saddle points. The separatices through A_i (which we denote by $Z_i^{\pm}(\eta)$) have slopes

$$\lambda^{\pm}(\theta_i) = \{-\mu \pm [\mu^2 + 4\nu H'(\theta_i)]^{\frac{1}{2}}\}/2\nu, \tag{3.16}$$

where $\lambda^{+}(\theta_i) > 0 > \lambda^{-}(\theta_i)$, since $H'(\theta_i) > 0$, for $i = 0, 2$. The slopes at B_1 are also given by (3.16), where $H'(\theta_1) < 0$. When $I(\theta_1) > 0$, where

$$I(\eta) = \mu^2 + 4\nu H'(\eta), \tag{3.17}$$

B_1 is a node, while if $I(\theta_1) < 0$, B_1 is a focal point. Thus for a given forcing function H (which is of $O(\epsilon)$) B_1 will be a node if there is sufficient damping in the system. It may then be possible to construct a continuous solution passing through A_0, B_1 and A_2 for any value of δ. Obviously the nodal condition $I(\theta_1) > 0$ is *necessary* for the existence of such a solution, however it may not be sufficient. Since the distortion of the signal, and possible shock formation, depends on the amplitude of $H'(\eta)$, one can expect the condition ensuring the existence of a continuous solution to depend on a global property of $H'(\eta)$. In fact

$$\mu^2 > \mu_c^2 = \max_{\eta}\,[-4\nu H'(\eta)] > 0 \tag{3.18}$$

is a *sufficient* condition for the existence of a continuous periodic solution at all frequencies. The proof of this result is given in the appendix. Thus for $\mu > \mu_c$, $F(\eta) = Z_0^{+}(\eta)$, which is continuous and, by (3.5), satisfies the mean condition (2.24) (see figure 1).

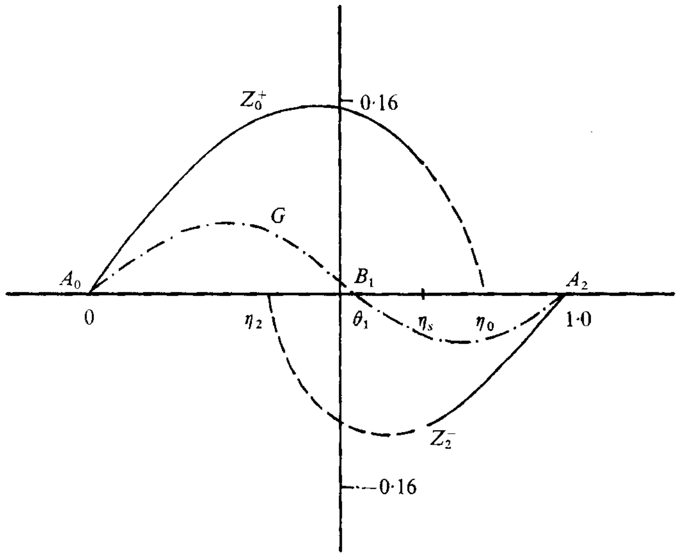

FIGURE 2. Construction of solution F using integral curves Z_0^+ and Z_2^-. Shock is at η_s. $\epsilon = 0{\cdot}0147$, $\delta = 0{\cdot}07$, $i = 0{\cdot}08$.

3.3. $0 < \mu < \mu_c$

In this case there is not enough damping to produce a shockless solution for all frequencies. Thus for $\delta = 0$

$$Z_0^+(\theta_1) > 0, \quad Z_2^-(\theta_1) < 0, \quad Z_0^+(\eta_0) = Z_2^-(\eta_2) = 0,$$

where $0 < \eta_2 < \theta_1 < \eta_0 < 1$ (see figure 2). Hence the separatices do not connect the saddle points A_0 and A_2. As δ is increased through the resonant band, so that

$$\int_0^1 G(s)\, ds > 0,$$

there exists a unique frequency, given by $\delta = \delta^+$, such that $Z_0^+(\eta) \equiv Z_2^+(\eta)$, $\theta_0 \leqslant \eta \leqslant \theta_2$. That is, for $\delta = \delta^+$, the positive separatrix connects the saddle points A_0 and A_2 (see figure 3). Further, for $\delta > \delta^+$ there exists a *unique*, continuous periodic solution $Z = Z_\delta(\eta) > 0$, (see figure 4). Similarly there exists a $\delta = \delta^- < 0$ for which $Z_0^- \equiv Z_2^-$ and such that when $\delta < \delta^-$ there is a *unique*, continuous periodic solution $Z = Z_\delta(\eta) < 0$. These results can be inferred from the results of Amerio (1949, 1950). Whereas for the case $\mu = 0$ explicit values have been given for $\delta^\pm$ (see equation (3.12)), when $0 < \mu < \mu_c$ this is not possible. However, for a particular forcing function, $\delta^\pm$ can easily be found numerically by varying δ until a solution is found such that $Z_0^+(\theta_2) = 0$ or $Z_2^-(\theta_0) = 0$. Since these limiting solutions are continuous they satisfy the mean condition (2.24) and hence we can give the implicit conditions for the edge of the band as

$$\int_{\theta_0}^{\theta_2} Z_0^+(s, \delta^+)\, ds = \frac{\delta^+}{b(\delta^+)}, \quad \int_{\theta_0}^{\theta_2} Z_2^-(s, \delta^-)\, ds = \frac{\delta^-}{b(\delta^-)}.$$

There have been several attempts to obtain analytical bounds on $\delta^\pm$. Hayes (1953) and Böhm (1953) found bounds for $h = -\cos 2\pi\eta$ while Lillo & Seifert (1955)

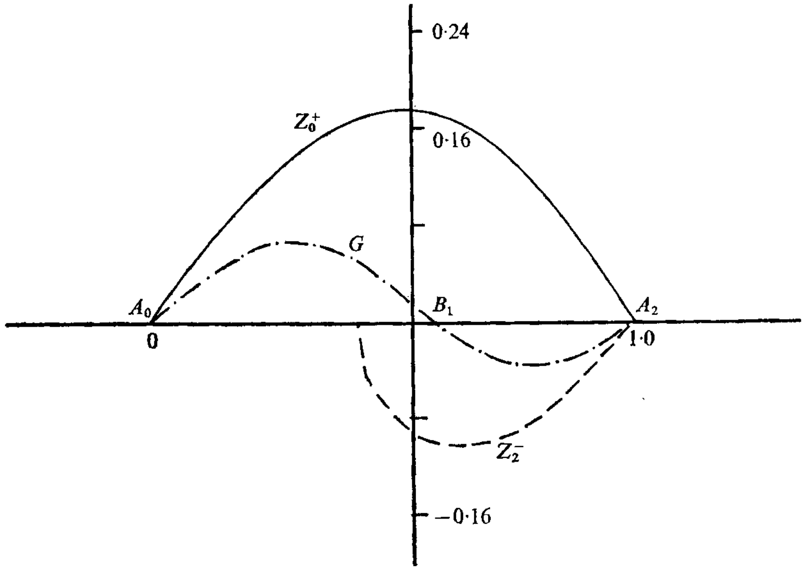

FIGURE 3. $F = Z_0^+$ is solution at edge of resonant band. $\epsilon = 0{\cdot}0147$, $\delta^+ = 0{\cdot}14$, $i = 0{\cdot}08$.

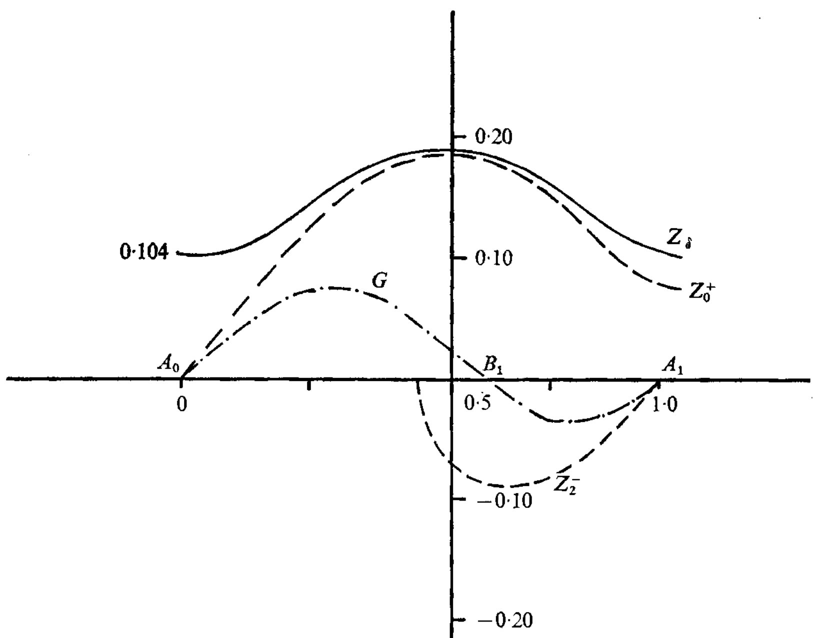

FIGURE 4. $F = Z_\delta$ is continuous solution outside resonant band. $\epsilon = 0{\cdot}0147$, $\delta = 0{\cdot}20$, $i = 0{\cdot}08$.

used similar techniques to find bounds for a general forcing function. Further reference can be found in Sansone & Conti (1964).

By equation (3.5) the *unique continuous* solutions $Z_\delta(\eta)$ automatically satisfy the mean condition (2.24) and hence, for $\delta > \delta^+$ or $\delta < \delta^-$,

$$F(\eta) = Z_\delta(\eta), \quad \theta_0 \leqslant \eta \leqslant \theta_2.$$

When $\delta^- < \delta < \delta^+$ there are no continuous periodic solutions of (3.15). Again, as in the case $\mu = 0$, we construct the signal function F by a composition of integral curves (see figure 2). The discontinuous signal function F must satisfy both the mean condition (2.24) and the weak-shock condition (3.4). However, it has been shown that if the mean condition is satisfied the shock condition is automatically satisfied. The condition that a shock is compressive then implies that we choose

$$F(\eta) = \begin{cases} Z_0^+(\eta), & \theta_0 \leqslant \eta < \eta_s, \\ Z_2^-(\eta), & \eta_s < \eta \leqslant \theta_2, \end{cases} \tag{3.19}$$

where $\eta = \eta_s$ is the position of the shock. It is shown in the appendix that it is always possible to choose an η_s to combine $Z_0^+(\eta)$ and $Z_2^-(\eta)$ so that the mean condition (2.24) is satisfied. Hence F, as given by (3.19), is the required signal function. By (3.15), $F' = 0$ at $\eta = \eta_{\max}, \eta_{\min}$, where $F = G/\mu$. Then, in general, when $\omega = \omega_n$ the maximum pressure exceeds the pressure immediately ahead of the shock and the pressure immediately behind the shock exceeds the minimum pressure.

It is clear from the structure of the integral curves for the undamped case that if the piston frequency is an even multiple of the fundamental, then a possible continuous solution is

$$F(\eta) = Z_0^+(\eta), \quad \theta_0 \leqslant \eta \leqslant \theta_2,$$

with

$$F(\eta+1) = -F(\eta),$$

which is a 'subharmonic' solution. From the preceding analysis, this solution is unstable to perturbations in both damping and frequency.

Finally we note that when $k = 0$ ($\mu = 1$) the impedances at $x = 0$ are 'matched'. Then there is no reflected wave, so that by (2.14) $g \equiv 0$. The differential equation (2.26), together with (2.19) and (2.23), then yields that, on $x = 1$,

$$f(\eta) = H(\eta) \quad \text{for all } \eta.$$

4. Critical acceleration level

In the preceding we have analysed resonant oscillations when it is assumed that the only damping in the system is due to radiation of energy away from the end of the pipe. Here we show that, in the high frequency limit, the effect of damping due to internal dissipation can also be incorporated into equation (2.26). While these may not be the only damping mechanisms present, they can be dominant. Then we give a condition on the piston acceleration, in terms of the physical parameters, which ensures a shockless motion.

It has been shown by Mortell & Seymour (1972*a*) that the representation (2.10) and (2.11) can be extended, in the high frequency limit, to include the effect of internal dissipation of the transmitting media (specifically, there, for a viscoelastic rod). For a gas such dissipation would result from the excitation of any of the internal degrees of freedom, e.g. vibrational excitation or molecular dissociation. If it is assumed that only one rate-dependent process is of significance and that this can be represented by the relaxation variable $\sigma(x, t)$, then the rate of adjustment of σ may be described by

$$\partial\sigma/\partial t = \psi(p, e, \sigma).$$

ψ will then define a rate parameter or relaxation time τ (> 0) proportional to $\psi_\sigma(p_0, e_0, \sigma_0)$. A periodic disturbance is considered of high frequency, or is near-frozen, if its period is small compared with τ, so that $L(a_0\omega)^{-1} \ll \tau$ (note that in our variables $L(a_0\omega)^{-1}$ represents the dimensional piston frequency). For such disturbances a_0 should now be interpreted as the frozen rather than equilibrium sound speed (more details of nonlinear wave propagation in a relaxing gas are given by Blythe (1969)). In this limit, a disturbance in the gas can be represented as two non-interacting modulated simple waves travelling in opposite directions (see Mortell & Seymour 1972*a*; modulated simple waves in rate-dependent media are discussed in detail by Seymour & Varley 1970). The appropriate representation corresponding to equations (2.10) and (2.11) is then

$$e = p = -f(\beta)\, e^{d(x-1)} - g(\alpha)\, e^{-dx}, \quad u = f(\beta)\, e^{d(x-1)} - g(\alpha)\, e^{-dx}, \tag{4.1}$$

$$\alpha/\omega = t - x - Mg(\alpha)\, d^{-1}(1 - e^{-dx}) \tag{4.2}$$

and

$$\beta/\omega = t + x - 1 + Mf(\beta)\, d^{-1}(e^{d(x-1)} - 1), \tag{4.3}$$

where $d = L/(a_0\tau) \ll \omega$. On eliminating g from (4.1)–(4.3) through the boundary conditions (2.4) and (2.5), the functional difference equation (2.12), and (2.13), is recovered with the parameters k and b replaced by $\bar{k}$ and $\bar{b}$, where

$$\bar{k} = ke^{-2d}, \quad \bar{b} = \tfrac{1}{2}M(1+\delta)\,(1 + ke^{-d})\,(1 - e^{-d})\, d^{-1}. \tag{4.4}$$

In the small rate limit, when $\omega = O(1)$, $d \ll 1$. Under these circumstances the procedure of § 2 then leads to the nonlinear ordinary differential equation (2.26) with the parameters μ and ν replaced by $\bar{\mu} = 1 - \bar{k}$ and $\bar{\nu} = n\bar{b}\bar{k}$. The rate-independent case is recovered in the limit $\tau \to \infty$ when $d \to 0$. The parameter $\bar{k}$ in (4.4) consists of two factors; the first, k, is the attenuation of the signal at the interface $x = 0$ due to transmission of energy into the neighbouring medium; the second, e^{-2d}, is the attenuation of the signal over one cycle due to internal damping. While the latter acts continuously throughout the gas, whereas the former only acts at the interface, they enter (2.26) in an identical manner. Thus the role of both in preventing shock formation is the same.

Betchov (1958) modelled the effect of boundary-layer friction on the flow by introducing a body-force term in the momentum equation proportional to the fluid velocity. This idea goes back to Rayleigh (see Rayleigh 1945). This model of the friction has the same effect on the flow as rate dependence, and again enters (2.26) through $\bar{\mu}$.

Since $A_p(t) = \omega H'(\omega t)$ represents the acceleration of the piston, we can interpret condition (3.18) as follows: for a given amount of damping there is a critical acceleration level of the piston such that for applied accelerations below this the motion of the gas is shockless at all frequencies. By (3.18), for a known $\bar{\mu}$ the gas motion is shockless provided that the *piston* acceleration satisfies

$$|A_p(t)| < \frac{\omega\bar{\mu}^2}{4\bar{\nu}} = \frac{\bar{\mu}^2}{4M\bar{k}(1+\bar{k})} \tag{4.5}$$

on using (2.17), (2.18) and (2.20). In the limit as the damping tends to zero, $\bar{k} \to 1$ and $\bar{\mu} \to 0$, which implies $|A_p(t)| \to 0$ for a shockless motion. Thus when there is no damping present there is always a shock at resonance.

When $i \ll 1$ and $a \ll 1$, then (4.5) reduces to

$$|MA_p(t)| < \tfrac{1}{2}(i+a)^2 \tag{4.6}$$

as the condition for a shockless solution for a resonant *forced* motion. In contrast, for a *transient* or 'standing wave' motion in the same system the condition for a shockless motion is (see Mortell & Seymour 1972*a*)

$$|MA(t)| < (i+a), \tag{4.7}$$

where $A(t)$ here is the acceleration level defined by the initial conditions. The formation of a shock is determined by the induced acceleration level in the gas flow. For the shockless transient motion the applied and induced accelerations have the same order of magnitude. However, for the resonant forced motion, the induced acceleration has the same order of magnitude as the *square root* of the applied acceleration. With this observation the results (4.6) and (4.7) are in harmony. In a theoretical study of resonant oscillations of a radiating gas, Eninger (1971) found, using numerical methods, that there was a critical value of the radiative parameter above which the motion remained shockless.

When the resonant motion is shockless, linear theory is a uniformly good approximation to the nonlinear theory provided that the piston acceleration is sufficiently small (see figure 1). By setting $\nu = 0$ in (2.26), the linear solution is given by

$$F_L(\eta) = G(\eta)/\mu$$

and hence, by (2.26),

$$\left|\frac{F_L(\eta)}{F(\eta)} - 1\right| = \frac{\nu}{\mu}|F'(\eta)|.$$

By differentiating (2.26) and setting $F''(\eta) = 0$ we find that $|F'(\eta)| \leqslant \lambda^+(\eta)$, where $\lambda^+(\eta)$ is given by (3.16). If now the applied acceleration is small, in the sense that

$$|4\nu H'(\eta)/\mu^2| \ll 1, \tag{4.8}$$

then

$$\begin{aligned}\left|\frac{F_L(\eta)}{F(\eta)} - 1\right| &= \frac{\nu}{\mu}\left|F'(\eta)\right| \leqslant \frac{\nu}{\mu}\left|\lambda^+(\eta)\right| \\ &= \frac{1}{2}\left|\left(1+\frac{4\nu H'}{\mu^2}\right)^{\frac{1}{2}} - 1\right| \\ &= \left|\frac{\nu}{\mu^2}H'\right| + O\left(\frac{\nu}{\mu^2}H'\right)^2 \\ &\ll 1, \quad \text{by (4.8)}.\end{aligned}$$

The inequality (4.5) defines a critical acceleration level which provides a sufficient condition on the applied rate to ensure a response of the gas which is continuous. Numerical integration of the equations shows that shockless solutions exist for piston accelerations greater than the critical one, so that inequality (4.5) is conservative, as may be expected.

5. Comparison with experiments

The theoretical predictions of the analysis presented are compared with some experimental measurements made by Sturtevant (private communication). His set up consists of a tube of length 132·5 in. with an inside diameter of 3·0 in. which contains air ($\gamma = 1{\cdot}4$). At one end of the tube is a piston which is displaced sinusoidally with an amplitude, normalized against the length of the tube, of 0·0147. The experiments we are concerned with here have the two configurations

(i) when the far end of the tube is closed;

(ii) when the far end of the tube has a hole in it whose area is small compared with the area of the end.

For case (i) we are concerned with measurements of the following quantities, at the closed end, for values of the piston frequency around the fundamental:

(*a*) the absolute maximum and minimum of a normalized pressure wave form,

(*b*) the pressure immediately before and after the shock jump.

As a consequence of these readings the values of the frequency corresponding to the lower and upper ends of the resonant band are available.

Figure 5(*a*) shows the comparison between the undamped case ($i = 0$) and experimental 'response curve' of Sturtevant. In this case, our theory is equivalent to that in § 4 of Chester (1964). In computing the theoretical curves in figure 5(*b*) a value of $i = (1-\bar{k})/(1+\bar{k})$ ($i = 0.08$) is chosen so that the shock *strength* exactly at resonance ($\delta = 0$) is equal to the observed strength. The theory predicts that, for $\delta = 0$, the pressure immediately before the shock is the negative of the pressure immediately afterwards, which is not the experimental result. Thus the theoretical and experimental curves do not coincide at $\delta = 0$, even with our choice of $\bar{k}$. We should also bear in mind that for the conditions of the experiment the small rate condition is only marginally satisfied (see the comment at the end of § 2). The experiments show that at resonance the maximum pressure exceeds the pressure ahead of the shock and the pressure behind the shock exceeds the minimum pressure. This is not predicted by inviscid theory, but is a property of the solution of the equation with damping. Another point to note is that the absolute maximum of the pressure occurs about 10 % to the right of the resonant frequency, while the absolute minimum occurs about 5 % to the left.

An interesting point is that the amount of damping required to correct the shock strength at $\delta = 0$ has a negligible effect on the width of the resonant band. This might seem surprising since damping decreases the shock strength, which in turn determines the resonant band. The result can be understood, qualitatively, by considering equation (2.26) and bearing in mind the definition of $G(\eta)$ given by (2.23). When there is damping, $\mu \neq 0$, the system defined by (2.26) is being driven by the forcing function G whose mean is non-zero for $\delta \neq 0$. The increased damping is then counteracted by the increased amplitude of the effective driver G.

For case (ii) it is observed that for particular experimental conditions there is a critical area ratio at which the shock in the tube disappears for all frequencies. If we interpret the presence of a small hole in the end of the tube as a means of introducing damping into the system then the prediction of the theory agrees

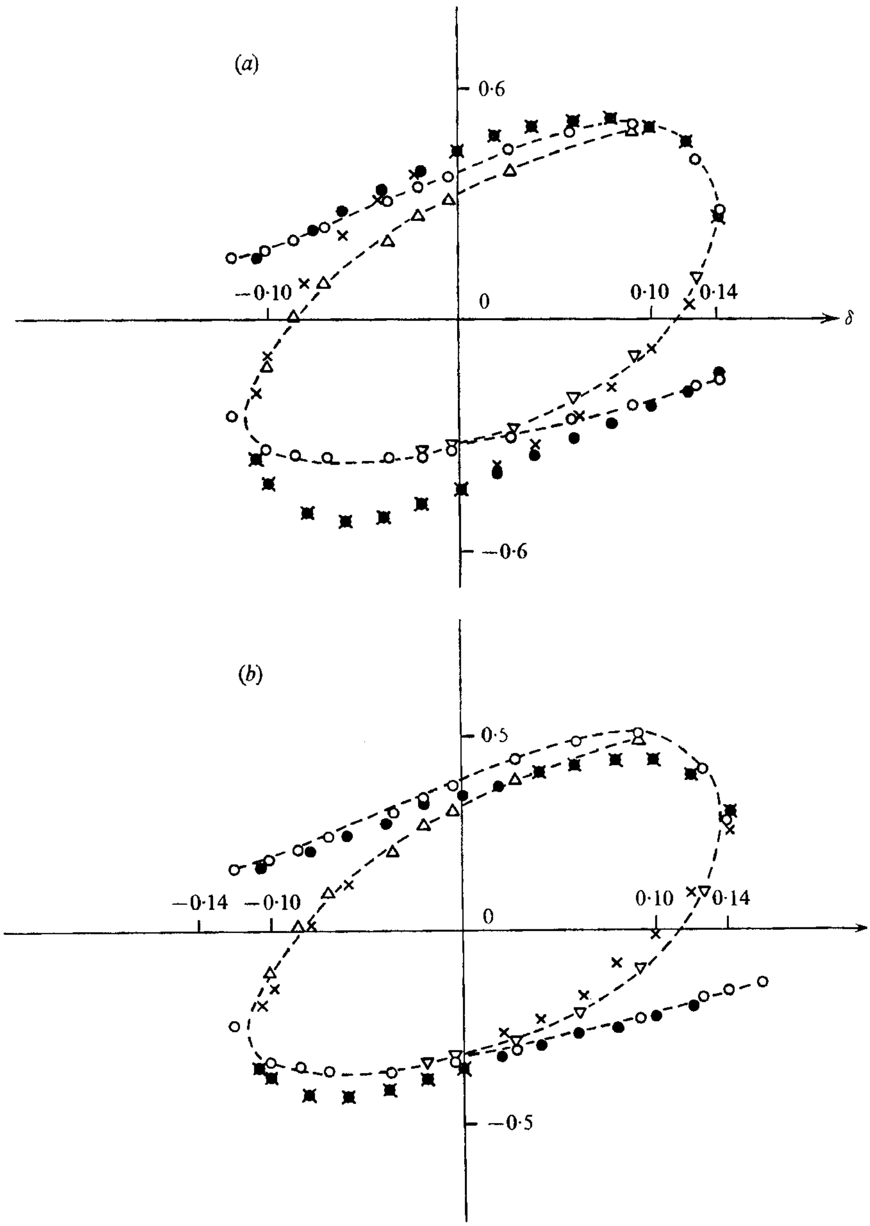

FIGURE 5. Theoretical and experimental response curves. (a) No damping, $i = 0$. (b) Damped case, $i = 0{\cdot}08$. $\epsilon = 0{\cdot}0147$, $\gamma = 1{\cdot}4$. Experiment: ○, maximum and minimum pressure; △, pressure before and after shock (due to Sturtevant). Theory: ●, maximum and minimum pressure; ×, pressure before and after shock.

qualitatively with experiment. It cannot be expected that the impedance condition (2.4), as introduced in the theory, will account for the detailed motion of the gas near the orifice. Nevertheless, it seems to be useful in predicting the gross features of the motion.

Curves of shock strength $S(i)$ versus impedance i were plotted for various

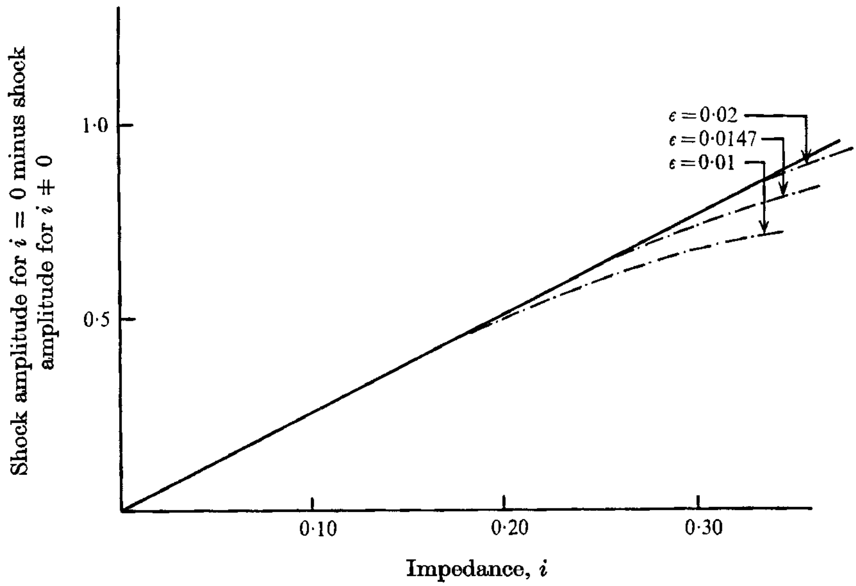

FIGURE 6. Shock amplitude versus impedance for various piston amplitudes.

values of the piston amplitude ϵ. Figure 6 shows that there is a linear relation between $S(0)-S(i)$ and i, which is independent of ϵ for $0 \leqslant i < 0{\cdot}2$ when $0{\cdot}01 \leqslant \epsilon \leqslant 0{\cdot}02$. A corresponding plot of shock strength versus area ratio would give a measure of the effective impedance (or effective damping). The linear relationship indicates that shock strength is a good measure of damping.

This result can be understood from a rough analysis of the energy balance. When $i \ll 1$, the results of § 3 indicate that if the amplitude of the piston is ϵ, the amplitude of the response is of $O(\epsilon^{\frac{1}{2}})$, while the shock strength S is of $O(\epsilon^{\frac{1}{2}})$. The balance between the input of energy due to the piston and the loss due to the shock and radiation from the end is

$$A\epsilon^{\frac{3}{2}} = S^3(i) + iB\epsilon, \tag{5.1}$$

where A and B are constants. Since $S = O(\epsilon^{\frac{1}{2}})$, equation (5.1) can be interpreted, dividing through by ϵ, as

$$S(i) - S(0) = iB',$$

where $S(0) = A'\epsilon^{\frac{1}{2}}$, and A' and B' are constants. The linear relationship is lost when shock dissipation is no longer a major effect. As the piston amplitude ϵ decreases, the point at which the curves bifurcate moves towards the origin, so that the linear relation holds for a smaller range of the impedance.

The results in this paper were obtained in the course of research sponsored in part by the U.S. Army under Contract No. DAAD05-71-C-0389 and monitored by the Ballistics Research Laboratories, Aberdeen Proving Ground, Md. and in part by the Air Force Office of Scientific Research under Grant No. AFOSR-71-2107.

Appendix

Here we prove two results used in § 3.

(i) A sufficient condition for the existence of a continuous periodic solution of (3.15), for all δ, is that

$$\mu^2 > \max_{\eta}[-4\nu H'(\eta)]. \tag{A 1}$$

(ii) Given $\mu \in (0, \mu_c)$, $\delta \in (\delta^-, \delta^+)$ and F defined by (3.19), there exists an $\eta_s \in (\eta_2, \eta_0)$ such that

$$\int_{\theta_0}^{\theta_2} F(s)\,ds = \frac{\delta}{b}. \tag{A 2}$$

Proof of (i). Using the notation of § 3 we must show that when (A 1) holds the separatices Z_0^+ and Z_2^- pass through the node B_1, i.e. $Z_0^+(\theta_1) = Z_2^-(\theta_1) = 0$. We shall prove the result for Z_0^+; the argument for Z_2^- is similar.

First, since the curve $G(\eta)/\mu$ is the isocline $Z'(\eta) = 0$ and $Z(\eta) \to 0^+$ yields the isocline $Z'(\eta) \to +\infty$, for $\theta_0 < \eta < \theta_1$, the separatrix Z_0^+ is continuous and differentiable in (θ_0, θ_1) and satisfies

$$0 < Z_0^+(\eta) \leqslant \max_{\eta}[G(\eta)/\mu].$$

In particular $Z_0^+(\theta_1) \geqslant 0$. We show that $Z_0^+(\theta_1) = 0$ by bounding $Z_0^+(\eta)$ above by a function $Y(\eta)$ which has the properties $Y(\eta) > 0$ for $\theta_0 < \eta < \theta_1$ and $Y(\theta_1) = 0$. Such a curve bounds Z_0^+ above if $dZ_0^+/d\eta < Y'(\eta)$, for all $\eta \epsilon [\theta_0, \theta_1]$, whenever $Z_0^+ = Y$. The curve $Y(\eta) = 2G(\eta)/\mu$ has these properties whenever (A 1) holds. For when $Z_0^+ = Y$,

$$\frac{dZ_0^+}{d\eta} = \frac{G-\mu Y}{\nu Y} = \frac{-\mu}{2\nu} < \frac{2H'(\eta)}{\mu} = Y',$$

which holds whenever condition (A 1) does. Hence since $Y(\theta_1) = 0$, $Z_0^+(\theta_1) = 0$.

Proof of (ii). Defining

$$y(\eta_s) = \int_{\theta_0}^{\theta_2} F(s)\,ds = \int_{\theta_0}^{\eta_s} Z_0^+(s)\,ds + \int_{\eta_s}^{\theta_2} Z_2^-(s)\,ds$$

we wish to show that, for a given δ, there is a value of $\eta_s \in (\eta_2, \eta_0)$ such that $y(\eta_s) = \delta/b$, where $\theta_0 < \eta_2 < \theta_1 < \eta_0 < \theta_2$, and $Z_0^+(\eta_0) = Z_2^-(\eta_2) = 0$. We first note that, for a given δ, y is a continuous function of η_s. Since Z_0^+ is continuous in (θ_0, η_0), integration of (3.15) yields

$$\int_{\theta_0}^{\eta_0} Z_0^+(s)\,ds = \frac{1}{\mu}\int_{\theta_0}^{\eta_0} G(s)\,ds.$$

Then consider

$$y(\eta_0) = \int_{\theta_0}^{\eta_0} Z_0^+(s)\,ds + \int_{\eta_0}^{\theta_2} Z_2^-(s)\,ds$$

$$= \frac{1}{\mu}\int_{\theta_0}^{\theta_2} G(s)\,ds + \int_{\eta_0}^{\theta_2}\left(Z_2^-(s) - \frac{G(s)}{\mu}\right)ds.$$

Thus, since $Z_2^-(\eta) \geqslant G(\eta)/\mu$ for $\theta_1 < \eta < \theta_2$,

$$y(\eta_0) \geqslant \frac{1}{\mu}\int_{\theta_0}^{\theta_2} G(s)\,ds = \frac{\delta}{b}.$$

Similarly $y(\eta_2) \leqslant \delta/b$. Therefore by the continuity of y, there is an $\eta_s \in (\eta_2, \eta_0)$ such that

$$y(\eta_s) = \int_{\theta_0}^{\theta_2} F(s)\,ds = \frac{\delta}{b}.$$

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